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Pressure-wave propagation through a separated gas–liquid layer in a duct

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Pressure-wave propagation through a separated gas–liquid layer at rest in a duct of constant rectangular cross-section and infinite length is considered. Such a system is dispersive, possessing an infinite number of modes which depend on the ratios of the densities, thicknesses and sound speeds of the two phases. The transitional variation of an infinitesimal disturbance initially having a step profile is investigated analytically and numerically. In addition, it is shown that a weak but finite disturbance is described asymptotically by the solution of the Korteweg–de Vries equation.

1. Introduction

In recent years considerable attention has been paid to fluid systems in which gaseous and liquid phases coexist. The dynamical properties of such fluid systems differ in general, depending on whether each phase is discrete or continuous, examples of the different possibilities being bubble–liquid, droplet–gas, separated stratified and plug systems. On the other hand, however, all these systems have the common feature that they admit dispersive propagating pressure waves. Thus they pose an interesting problem in connexion with wave motion or choking of the flow.

For bubble–liquid systems, in which the liquid phase is continuous but the gaseous phase is discrete, we can refer to van Wijngaarden's (1968, 1972) model; the analysis is based on this model. Droplet–gas systems, in which the liquid phase is discrete but the gaseous phase is continuous, have been investigated extensively, as an example of a dusty gas and, more comprehensively, as an example of a relaxing gas (e.g. Rudinger 1969). We can also find the dispersive properties of separated stratified gas–liquid systems, in which each phase is continuous (Morioka & Matsui 1973), and plug systems, in which both phases are discrete (Matsui & Morioka 1974).

In this paper, we consider pressure-wave propagation through a simple separated gas–liquid layer at rest in a duct of constant rectangular section and infinite length (see figure 1). It is shown that this system is dispersive, possessing multiple (infinitely many) modes which depend only on the ratios of the densities, thicknesses and sound speeds of the two phases in spite of the two-dimensional configuration (§2). We consider the wave motion generated by suddenly removing a diaphragm dividing the duct into two chambers in which the pressures are

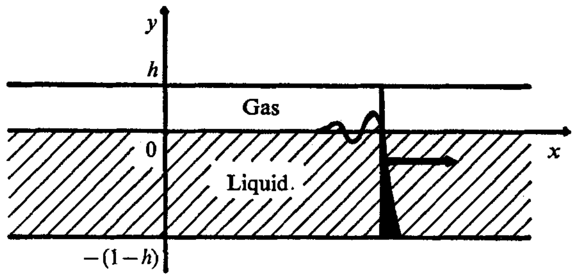


FIGURE 1. Sketch of pressure-wave propagation through a separated gas–liquid layer at rest in a duct of constant rectangular cross-section and infinite length.

different. Since the dispersion relation is presented in the form of a transcendental equation, it is difficult to express the solution explicitly, even if we restrict ourselves to infinitesimal waves. Here the formal solution uniformly valid within the linearized theory is presented by using the Laplace–Fourier transform technique (§3). By integrating numerically such a formal solution, we can illustrate the characteristic variations in the pressure distribution along the duct wall and the shape of the interface, and also their dependence on the ratio of the densities of the two phases (§4). These numerical results show that the pressure field approaches a one-dimensional configuration moving at a constant speed as time progresses. This tendency, as well as the behaviour of the phase velocity in the low frequency limit, suggests that the effect of weak nonlinearity may be taken into account by applying the reductive perturbation method (Gardner & Morikawa 1960; Taniuti & Wei 1968). In fact it is shown that the pressure field is described asymptotically by the Korteweg–de Vries equation (§5).

2. Dispersion relation

The motion is assumed to be two-dimensional and the effects of gravity and viscosity are neglected. Then the equations describing the motion may be written, in non-dimensional form, as follows:

$$\frac{\partial \rho_i}{\partial t}+\frac{\partial}{\partial x}(\rho_i U_i)+\frac{\partial}{\partial y}(\rho_i V_i) = 0, \qquad (1a)$$

$$\frac{\partial U_i}{\partial t}+U_i\frac{\partial U_i}{\partial x}+V_i\frac{\partial U_i}{\partial y} = -\frac{1}{\gamma\rho_*\rho_i}\frac{\partial P_i}{\partial x}, \qquad (1b)$$

$$\frac{\partial V_i}{\partial t}+U_i\frac{\partial V_i}{\partial x}+V_i\frac{\partial V_i}{\partial y} = -\frac{1}{\gamma\rho_*\rho_i}\frac{\partial P_i}{\partial y}, \qquad (1c)$$

$$dP_i/P_i = \gamma\rho_* c_*^2\, d\rho_i/\rho_i. \qquad (1d)$$

Here the suffix i must be replaced by g and l for the gaseous and liquid phases respectively. ρ_* is the ratio of the liquid density and the gas density and c_* is the ratio of the sound speed in the liquid phase to that in the gaseous phase; they must be replaced by unity in the equations describing the gaseous phase. The length and time scales are L and Lc^{-1}, and the velocity components U_i and V_i, pressure P_i and density ρ_i have been made non-dimensional using c, P and $\rho_* \rho$, respectively, where L is the duct height and c, P and ρ are the sound speed, pressure and density of the gaseous phase at rest. γ denotes the ratio of the specific heats of the gas.

The boundary conditions for the fluid motion are derived from the requirements that the fluid flows along the duct wall and the interface is a tangential discontinuity:

$$\left.\begin{array}{l} V_g = 0 \quad \text{at} \quad y = h, \qquad V_l = 0 \quad \text{at} \quad y = -(1-h), \\ P_g = P_l, \quad V_i = \partial Y/\partial t + U_i \partial Y/\partial x \quad \text{at} \quad y = Y(t,x), \end{array}\right\} \tag{2}$$

where h (< 1) and $-(1-h)$ are the non-dimensional locations of the walls on the gaseous and liquid sides measured from the interface level in the rest state. The function $Y(t,x)$ denotes the shape of the interface and satisfies the initial condition $Y(0,x) = 0$.

Writing $U_i = \epsilon u_i$, $V_i = \epsilon v_i$, $P_i = 1+\epsilon p_i$, $\rho_i = 1+\epsilon \rho_i'$ and $Y = \epsilon Y'$, where $\epsilon \ll 1$, and linearizing (1) and (2), we have

$$\left.\begin{array}{c} \dfrac{1}{\gamma \rho_* c_*^2} \dfrac{\partial p_i}{\partial t} + \dfrac{\partial u_i}{\partial x} + \dfrac{\partial v_i}{\partial y} = 0, \\ \dfrac{\partial u_i}{\partial t} = -\dfrac{1}{\gamma \rho_*} \dfrac{\partial p_i}{\partial x}, \quad \dfrac{\partial v_i}{\partial t} = -\dfrac{1}{\gamma \rho_*} \dfrac{\partial p_i}{\partial y}, \end{array}\right\} \tag{3}$$

$$\left.\begin{array}{l} v_g = 0 \quad \text{at} \quad y = h, \qquad v_l = 0 \quad \text{at} \quad y = -(1-h), \\ p_g = p_l, \quad v_g = v_l = \partial Y'/\partial t \quad \text{at} \quad y = 0. \end{array}\right\} \tag{4}$$

Here ρ_i' has been eliminated by using the relation $\rho_i' = \gamma \rho_* c_*^2 p_i$.

Now we consider a simple harmonic disturbance of the form

$$u_i/\hat{u}_i(y) = v_i/\hat{v}_i(y) = p_i/\hat{p}_i(y) = e^{i(kx-\omega t)}. \tag{5}$$

Substituting these expressions into (3) and taking account of (4), we then find the dispersion relation

$$\lambda \tanh(h\lambda) + \rho_*^{-1} \lambda_* \tanh\{(1-h)\lambda_*\} = 0, \tag{6}$$

where $\lambda^2 = k^2 - \omega^2$ and $\lambda_*^2 = k^2 - \omega^2/c_*^2$. Equation (6) is consistent with the relation which determines the poles of the integrand in the formal solution discussed in the following section. Unfortunately we cannot express the roots of this transcendental equation explicitly, and must infer properties from several asymptotic expressions and numerical solutions.

We first obtain asymptotic expressions for large ρ_*:

$$\omega = \left\{\begin{array}{ll} k + \rho_*^{-1} \mu \tanh\{(1-h)\mu k\}/2h + \dots & (n = 0), \\ \{k^2 + (n\pi/h)^2\}^{\frac{1}{2}} + \dots & (n = 1, 2, \dots), \end{array}\right\} \tag{7}$$

where $\mu^2 = 1 - c_*^{-2}$ and n denotes the mode number. Then corresponding asymptotic expressions for the phase velocity ω/k and group velocity $\partial\omega/\partial k$ can be found. Thus a disturbance propagating through a separated gas–liquid system may consist of an infinite number of modes. Each phase velocity is larger than that

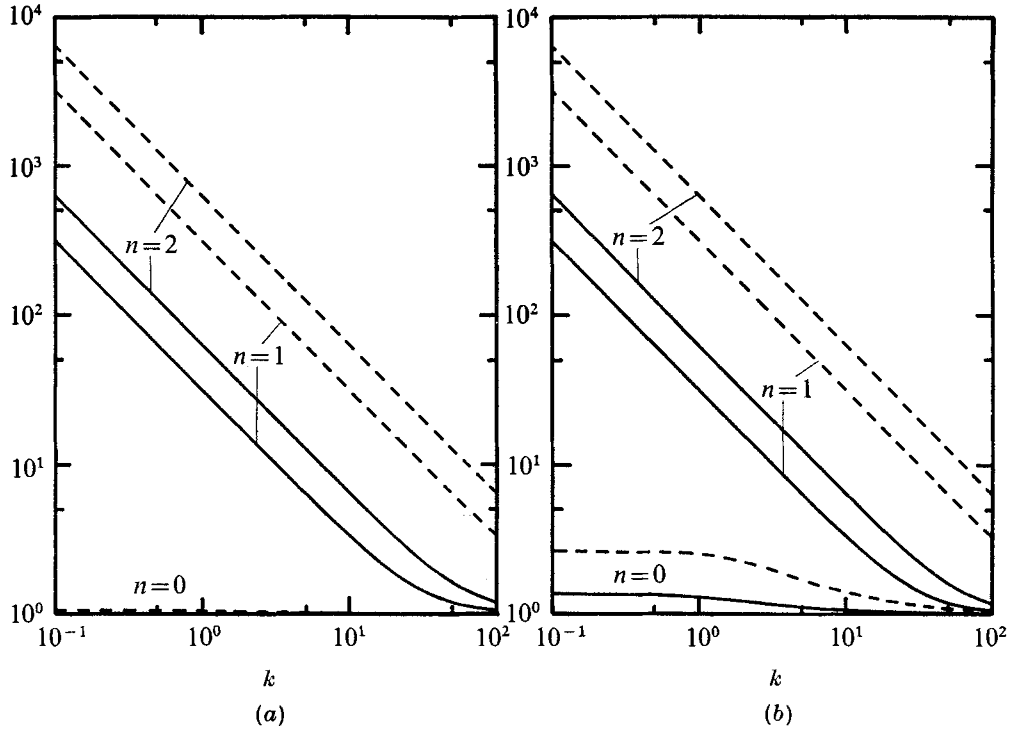


FIGURE 2. Numerical solution of the dispersion relation: (a) $\rho_* = 1000$ and (b) $\rho_* = 10$ with $c_* = 4$ and $h = 0{\cdot}1$ (solid line) or $0{\cdot}01$ (broken line). Phase velocities of the first three modes ($n = 0$, 1 and 2) are plotted *vs.* k.

in the gaseous phase alone and increases for lower wavenumbers and also higher mode numbers. The group velocity is smaller than the phase velocity for each mode and is smaller than that in the gaseous phase alone except for the zeroth mode. When ρ_*, h and c_*^{-2} are comparatively small, the system exhibits strong dispersion in the sense that the phase and group velocities vary considerably with frequency.

Another asymptotic expression can be obtained for the zeroth mode, for small k:

$$\omega^2 = k^2\left\{\frac{h+(1-h)/\rho_*}{h+(1-h)/\rho_* c_*^2} - \frac{\mu^2 h^2(1-h)^2}{3\rho_*}\frac{(1-h+h/\rho_*)}{[h+(1-h)/\rho_* c_*^2]^3}k^2+\ldots\right\}. \tag{8}$$

This form suggests that weak nonlinearity may be considered by applying the reductive perturbation method (Washimi & Taniuti 1966; Jeffrey & Kakutani 1972).

The numerical solution of the dispersion relation (6) is shown in figure 2 for several values of h and ρ_* and $c_* = 4$. Only the first three modes are shown. The iterative procedure, by Newton's method, easily converges starting from the value given by (7). The selected values of c_* and ρ_* correspond to an air–water system; $\rho_* = 1000$ corresponds to atmospheric conditions and $\rho_* = 10$ to high-pressure conditions. The results confirm the previous discussion based on the asymptotic representation.

3. Integral representation of the disturbance

As the initial state we assume a step pressure profile:

$$p_i = -\tfrac{1}{2}\operatorname{sgn} x, \quad u_i = v_i = 0 \quad \text{at} \quad t = 0. \tag{9}$$

We use Laplace transforms with respect to t and Fourier transforms with respect to x, e.g.

$$\bar{p}_i(s,k,y) = \int_{-\infty}^{\infty} e^{-ikx}\left\{\int_0^\infty e^{-st}p_i(t,x,y)\,dt\right\}dx,$$

and take account of (9). Then, (3) and (4) are reduced to simultaneous ordinary differential equations with respect to y:

$$\left(k^2+\frac{s^2}{c_*^2}\right)\bar{p}_i + \gamma\rho_* s\frac{d\bar{v}_i}{dy} = -\frac{s}{c_*^2 ik}, \tag{10a}$$

$$\gamma\rho_* s\bar{v}_i = -d\bar{p}_i/dy, \tag{10b}$$

$$\bar{v}_g = 0 \quad \text{at} \quad y = h, \qquad \bar{v}_l = 0 \quad \text{at} \quad y = -(1-h), \tag{11}$$

$$\bar{p}_g = \bar{p}_l, \quad \bar{v}_g = \bar{v}_l = s\bar{Y}' \quad \text{at} \quad y = 0.$$

Here the suffix i must be again replaced by g and l for the gaseous and liquid phases, respectively, and ρ_* and c_* must be replaced by unity in the equations for the gaseous phase. In deriving (10a), we have eliminated u_i by using the relation $\gamma\rho_* s\bar{u}_i = -ik\bar{p}_i$.

The boundary-value problem presented by (10) and (11) leads to the following solution:

$$\bar{p}_g = -\frac{s}{ik\lambda^2} - \frac{(s/ik)(c_*^{-2}\lambda_*^{-2}-\lambda^{-2})}{1+\rho_*\lambda\tanh(h\lambda)/\lambda_*\tanh\{(1-h)\lambda_*\}}\,\frac{\cosh\{(h-y)\lambda\}}{\cosh(h\lambda)}, \tag{12}$$

$$\bar{p}_l = -\frac{s}{ikc_*^2\lambda_*^2} + \frac{(s/ik)(c_*^{-2}\lambda_*^{-2}-\lambda^{-2})}{1+\rho_*^{-1}\lambda_*\tanh\{(1-h)\lambda_*\}/\lambda\tanh(h\lambda)}\,\frac{\cosh\{(1-h+y)\lambda_*\}}{\cosh\{(1-h)\lambda_*\}}, \tag{13}$$

$$\bar{v}_i|_{y=0} = -\frac{(\lambda/\gamma ik)(c_*^{-2}\lambda_*^{-2}-\lambda^{-2})}{1+\rho_*\lambda\tanh(h\lambda)/\lambda_*\tanh\{(1-h)\lambda_*\}}\tanh(h\lambda), \tag{14}$$

where $\lambda^2 = k^2+s^2$ and $\lambda_*^2 = k^2+s^2/c_*^2$. The pressure field can then be expressed as

$$\begin{aligned} p_i &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left\{\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\bar{p}_i e^{st}\,ds\right\}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left\{\sum_{n=0}\operatorname{Res}[\bar{p}_i e^{st}]_{s=i\omega_n}\right\}dk \equiv \sum_{n=0} p_i^{(n)}, \end{aligned} \tag{15}$$

according to the inversion formula. Here $\operatorname{Res}[\bar{p}_i e^{st}]_{s=i\omega_n}$ denotes the residue at the pole $s = i\omega_n$ in the integrand $\bar{p}_i e^{st}$, where ω_n is a real function of k provided by the dispersion relation (6). Thus $p_i^{(n)}$ represents the contribution from the nth mode. Calculating the residues by using (12) and (13), we have

$$p_g^{(n)} = \frac{2}{\pi}\int_0^\infty \frac{-(\nu^2+c_*^2\lambda_*^2)\dfrac{\cos\{(h-y)\nu\}}{\cos(h\nu)}\sin(xk)\cos(t\omega_n)}{\nu^2\{1+2(1-h)\lambda_*/\sinh[2(1-h)\lambda_*]\}+c_*^2\lambda_*^2\{1+2h\nu/\sin(2h\nu)\}}\frac{dk}{k}, \tag{16}$$

$$p_l^{(n)} = \frac{2}{\pi}\int_0^\infty \frac{-(\nu^2+c_*^2\lambda_*^2)\dfrac{\cosh\{(1-h+y)\lambda_*\}}{\cosh\{(1-h)\lambda_*\}}\sin(xk)\cos(t\omega_n)}{\nu^2\{1+2(1-h)\lambda_*/\sinh[2(1-h)\lambda_*]\}+c_*^2\lambda_*^2\{1+2h\nu/\sin(2h\nu)\}}\frac{dk}{k}, \tag{17}$$

where $\nu^2 = -\lambda^2$. For the higher modes ($n > 1$), λ_*^2 becomes negative for small k and then the real expression can be obtained by setting $\lambda_*^2 = -\nu_*^2$.

On the other hand, the shape of the interface between the gaseous and the liquid phases can be expressed as

$$\begin{aligned} Y' &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left\{\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{1}{s}\bar{v}_i|_{y=0}e^{st}\,ds\right\}dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left\{\operatorname{Res}\left[\frac{1}{s}\bar{v}_i|_{y=0}\,e^{st}\right]_{s=0} + \sum_{n=0}\operatorname{Res}\left[\frac{1}{s}\bar{v}_i|_{y=0}\,e^{st}\right]_{s=i\omega_n}\right\}dk \\ &\equiv Y_0 + \sum_{n=0} Y^{(n)}. \end{aligned} \tag{18}$$

Here $Y^{(n)}$ denotes the contribution from the nth mode. After calculating the residues by using (14), we have

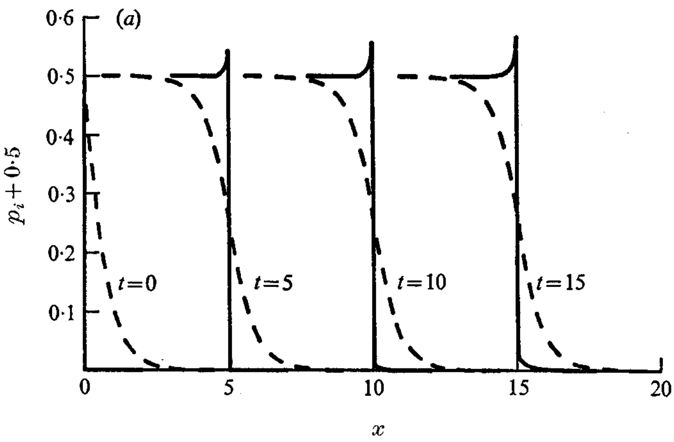
$$Y_0 = \frac{\mu^2}{\gamma\pi}\int_0^{\infty}\frac{\tanh(hk)\sin(xk)}{1+\rho_*\tanh(hk)/\tanh\{(1-h)k\}}\frac{dk}{k^2}, \tag{19}$$

$$Y^{(n)} = \frac{2}{\pi}\int_0^{\infty}\frac{-(\nu^2+c_*^2\lambda_*^2)\,\nu\tan(h\nu)\sin(xk)\cos(t\omega_n)}{\nu^2\{1+2(1-h)\lambda_*/\sinh[2(1-h)\lambda_*]\}+c_*^2\lambda_*^2\{1+2h\nu/\sin(2h\nu)\}}\frac{dk}{\gamma\omega_n^2 k}. \tag{20}$$

4. Numerical results

The integral expressions for the pressure field, (15)–(17), and the interface, (18)–(20), can be evaluated numerically using a computer. The calculation was performed for $\rho_* = 1000$ and 10 with $c_* = 4$ and $h = 0{\cdot}1$. These values of ρ_* and c_* correspond to an air–water system and the two values of ρ_* may be expected under atmospheric and high-pressure conditions, respectively. The pressure distributions along the upper and lower walls of the duct and the shape of the interface were represented at several time intervals after the motion had started. The calculation was based on numerical quadrature of the Fourier integrals by the ordinary trapezoidal rule, including the iterative solution of the transcendental dispersion equation. The quadrature points were at equal intervals $k = 0{\cdot}01$. This rule is adequate for the moderate values of x and t considered here, but Filon's (1928) formula taking into account the sinusoidal behaviour of the integrand may be used for larger values of x and t. For the pressure distribution along the lower (liquid side) wall, only the contribution from $k = 0$–20 was considered, since the integrand includes an effectively operating exponential term. On the other hand, for the upper (gaseous side) wall, the integrand executes a slow algebraically damped oscillation, and more quadrature points were required for convergence. Thus the contribution from $k = 0$–50 was considered. In the calculation of the interface shape, we took into account the contribution from $k = 0$–50. To examine the convergence of the results a check was carried out by comparing with the result for half the interval of integration. In either case the first three modes were taken into account, though the higher modes considered here contribute little.

Figure 3 (a) shows the pressure distributions along the upper and lower walls at the non-dimensional times $t = 0, 5, 10$ and 15 for the case $\rho_* = 1000, c_* = 4$ and



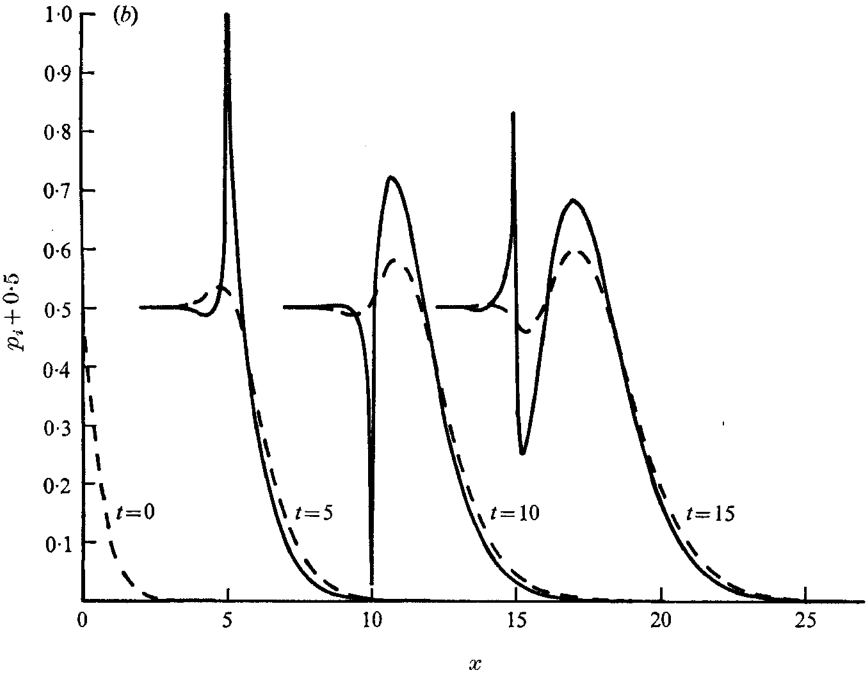


FIGURE 3. Pressure distribution along the duct wall at $t = 0$, 5, 10 and 15 for (*a*) $\rho_* = 1000$ and (*b*) $\rho_* = 10$ with $c_* = 4$ and $h = 0·1$. ——, upper (gaseous side) wall; ---, lower (liquid side) wall.

$h = 0·1$. The pressure distribution along the upper wall is almost a step profile moving with the sound speed of the gaseous phase, but a growing deviation may be observed in the neighbourhood of the corners. The pressure distribution along the lower wall exhibits a similar step profile but the corners are appreciably smoothed. The pressure profile on the lower wall does not exhibit the correct initial shape. This seems to be due to the neglect of the higher modes, which play an impoıtant role in the initial stage, especially in the liquid phase.

The initial pressure field can be described conveniently by making use of the

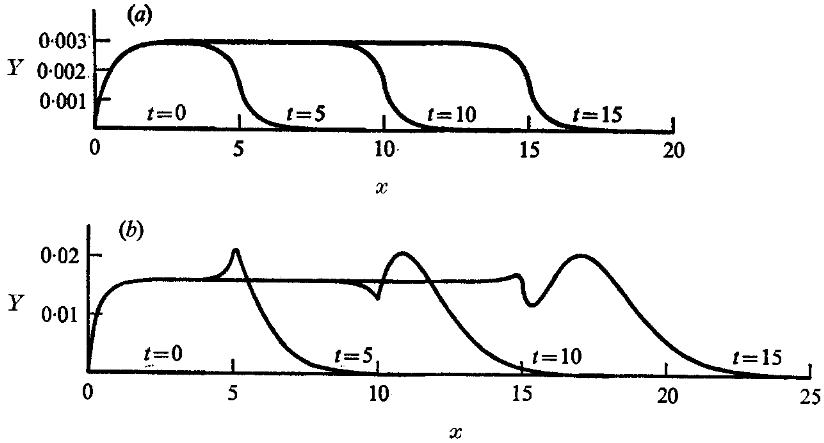


FIGURE 4. Shape of interface at $t = 0$, 5, 10 and 15, for (a) $\rho_* = 1000$ and (b) $\rho_* = 10$ with $c_* = 4$ and $h = 0{\cdot}1$.

wave expansion technique, as used by Cagniard (1939) in association with seismic waves. Since the neighbourhood of the starting point ($t = 0$, $x = 0$) corresponds to large values of s and k, the right-hand side of (13) with $y = -(1-h)$ may be expanded in powers of $\exp[-(1-h)\lambda_*]$ and $\exp(-h\lambda)$:

$$\bar{p}_l|_{y=-(1-h)} = -\frac{s}{ikc_*^2\lambda_*^2} + f_1(s,k)\exp[-(1-h)\lambda_*] + f_2(s,k)\exp[-2h\lambda-(1-h)\lambda_*] + f_3(s,k)\exp[-3(1-h)\lambda_*] + \ldots, \tag{21}$$

where f_1, f_2 and f_3 are algebraic functions of s and k. Inversion of the first term clearly gives

$$-\tfrac{1}{4}\operatorname{sgn}(x+c_*t) - \tfrac{1}{4}\operatorname{sgn}(x-c_*t), \tag{22}$$

while the subsequent terms contribute only for $t > (1-h)/c_*$, $2h-(1-h)/c_*$, $3(1-h)/c_*$ and so on and denote the direct and successively reflected effects of the interface interaction. Although the explicit representation of their inversions is difficult, it is clear that the pressure distribution on the lower wall is given by (22) for $t < (1-h)/c_*$ and satisfies the prescribed initial condition. Thus, our numerical calculation may be effective for $t \gg (1-h)/c_*$.

Figure 3 (b) shows the pressure distributions along the upper and lower walls of the duct for the case $\rho_* = 10$, $c_* = 4$ and $h = 0{\cdot}1$. An obvious wavy pattern appears in front of the near step profile moving with the sound speed of the gaseous phase. The pressure on the gaseous side abruptly fluctuates at the front, progressing with the sound speed of the gaseous phase. This change tends to induce an upstream wave and corresponding motion in the liquid phase. This is conspicuous at the initial stage but in the course of time approach to a one-dimensional configuration may be observed.

Figures 4 (a) and (b) show the shape of the interface at $t = 0$, 5, 10 and 15 for $\rho_* = 1000$ and 10, respectively. We can see a rise in the liquid phase in the compression region. This phenomenon, as well as its order of magnitude, will be rederived by the reductive perturbation procedure developed in the following section [see (32)].

5. Effect of weak nonlinearity

The numerical results in the preceding section show that the pressure field approaches a one-dimensional configuration moving with constant speed as time progresses. In addition, the asymptotic expression for the phase velocity for small k [§2, equation (3)] suggests that if we balance the weak nonlinearity and dispersion by applying the reductive perturbation method, the original system of equations may be described asymptotically by the Korteweg–de Vries equation. In this section we show that it is in fact so.

Now we return to the original nonlinear equations (1) and (2). We take the stretched co-ordinates

$$\tau = \epsilon^{\frac{3}{2}}t, \quad \xi = \epsilon^{\frac{1}{2}}(x/V - t), \quad y = y \tag{23}$$

and expand the perturbations in ascending powers of ϵ:

$$\left.\begin{aligned} U_i &= \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \dots, \\ V_i &= \epsilon^{\frac{1}{2}}(\epsilon v_i^{(1)} + \epsilon^2 v_i^{(2)} + \dots), \\ P_i &= 1 + \epsilon p_i^{(1)} + \epsilon^2 p_i^{(2)} + \dots, \\ \rho_i &= 1 + \epsilon \rho_i^{(1)} + \epsilon^2 \rho_i^{(2)} + \dots . \end{aligned}\right\} \tag{24}$$

Here V corresponds to the non-dimensional asymptotic phase velocity, which will be determined by the consistency relation for the system.

Substituting (23) and (24) into (1), and writing down the equations arising from the terms of lowest and second lowest order in ϵ, we obtain

$$-\frac{\partial \rho_i^{(1)}}{\partial \xi} + \frac{1}{V}\frac{\partial u_i^{(1)}}{\partial \xi} + \frac{\partial v_i^{(1)}}{\partial y} = 0, \tag{25a}$$

$$-\frac{\partial u_i^{(1)}}{\partial \xi} + \frac{1}{\gamma \rho_* V}\frac{\partial p_i^{(1)}}{\partial \xi} = 0, \tag{25b}$$

$$\partial p_i^{(1)}/\partial y = 0, \tag{25c}$$

$$dp_i^{(1)} - \gamma \rho_* c_*^2\, d\rho_i^{(1)} = 0, \tag{25d}$$

$$-\frac{\partial \rho_i^{(2)}}{\partial \xi} + \frac{1}{V}\frac{\partial u_i^{(2)}}{\partial \xi} + \frac{\partial v_i^{(2)}}{\partial y} = -\frac{\partial \rho_i^{(1)}}{\partial \tau} - \frac{1}{V}\frac{\partial}{\partial \xi}(\rho_i^{(1)} u_i^{(1)}) - \frac{\partial}{\partial y}(\rho_i^{(1)} v_i^{(1)}), \tag{26a}$$

$$-\frac{\partial u_i^{(2)}}{\partial \xi} + \frac{1}{\gamma \rho_* V}\frac{\partial p_i^{(2)}}{\partial \xi} = -\frac{\partial u_i^{(1)}}{\partial \tau} - \frac{1}{V} u_i^{(1)}\frac{\partial u_i^{(1)}}{\partial \xi} - v_i^{(1)}\frac{\partial u_i^{(1)}}{\partial y} + \frac{1}{\gamma \rho_* V}\rho_i^{(1)}\frac{\partial p_i^{(1)}}{\partial \xi}, \tag{26b}$$

$$\frac{1}{\gamma \rho_*}\frac{\partial p_i^{(2)}}{\partial y} = \frac{\partial v_i^{(1)}}{\partial \xi} + \frac{1}{\gamma \rho_*}\rho_i^{(1)}\frac{\partial p_i^{(1)}}{\partial y}, \tag{26c}$$

$$dp_i^{(2)} - \gamma \rho_* c_*^2\, d\rho_i^{(2)} = p_i^{(1)}\, dp_i^{(1)} - \gamma \rho_* c_*^2 \rho_i^{(1)}\, d\rho_i^{(1)}, \tag{26d}$$

where the suffix i must be replaced by g and l for the gaseous and liquid phases, respectively, and ρ_* and c_* by unity in the equations for the gaseous phase.

The boundary conditions can be obtained by substituting (23), (24) and

$$Y(\tau, \xi) = \epsilon Y^{(1)} + \epsilon^2 Y^{(2)} + \dots$$

into (2) and collecting the terms of lowest and second lowest order in ϵ:

$$v_g^{(1)} = 0 \quad \text{at} \quad y = h, \qquad v_l^{(1)} = 0 \quad \text{at} \quad y = -(1-h), \tag{27a}$$

$$p_g^{(1)} = p_l^{(1)}, \quad v_g^{(1)} = v_l^{(1)} = -\partial Y^{(1)}/\partial\xi \quad \text{at} \quad y = 0, \tag{27b}$$

$$v_g^{(2)} = 0 \quad \text{at} \quad y = h, \qquad v_l^{(2)} = 0 \quad \text{at} \quad y = -(1-h), \tag{28a}$$

$$p_g^{(2)} = p_l^{(2)}, \quad v_g^{(2)} + V^{-1}u_g^{(1)}v_g^{(1)} = v_l^{(2)} + V^{-1}u_l^{(1)}v_l^{(1)} \quad \text{at} \quad y = 0. \tag{28b}$$

In deriving the conditions for $y = 0$, we have used the expansion

$$U_i(\tau, \xi, Y) = U_i|_{y=0} + Y[\partial U_i/\partial y]_{y=0} + \dots \quad \text{etc.}$$

The solution to (25) satisfying (27 a) is found to be

$$\gamma V u_g^{(1)} = \gamma \rho_g^{(1)} = p_g^{(1)}(\tau, \xi), \tag{29a}$$

$$v_g^{(1)} = -\frac{1}{\gamma}\left(1 - \frac{1}{V^2}\right)(h-y)\frac{\partial p_g^{(1)}}{\partial \xi}, \tag{29b}$$

$$\gamma \rho_* V u_l^{(1)} = \gamma \rho_* c_*^2 \rho_l^{(1)} = p_l^{(1)}(\tau, \xi), \tag{29c}$$

$$v_l^{(1)} = -\frac{1}{\gamma\rho_*}\left(\frac{1}{V^2} - \frac{1}{c_*^2}\right)(1-h+y)\frac{\partial p_l^{(1)}}{\partial \xi}. \tag{29d}$$

In order to satisfy (27 b) we must take

$$p_g^{(1)}(\tau, \xi) = p_l^{(1)}(\tau, \xi) \equiv p^{(1)}(\tau, \xi), \tag{30}$$

$$V^2 = \left(h + \frac{1-h}{\rho_*}\right) \bigg/ \left(h + \frac{1-h}{\rho_* c_*^2}\right). \tag{31}$$

Thus the leading term in each expansion, except for the y component of the velocity, is independent of y. Explicitly, the asymptotic phase velocity V agrees with the leading term of (8). From the last relation of (27 b), we have

$$Y^{(1)} = \frac{\mu^2 h(1-h)}{\gamma\rho_*[h + (1-h)/\rho_*]} p^{(1)}(\tau, \xi). \tag{32}$$

The interface rises a distance of order $\mu^2 h(1-h)/(1-h+\rho_* h)$ in the compressed region. The quantity $p^{(1)}(\tau, \xi)$ is determined by the requirement that the boundary-value problem at the next order constitutes a consistent system.

When we substitute (29)–(31) into the right-hand sides of (26), they can be expressed in terms of $p^{(1)}(\tau, \xi)$ only. Integrating these equations and applying the condition (28 a), we obtain the following results:

$$p_g^{(2)} = -\left(1 - \frac{1}{V^2}\right)\left(hy - \frac{y^2}{2}\right)\frac{\partial^2 p^{(1)}}{\partial \xi^2} + F_g(\tau, \xi), \tag{33a}$$

$$\gamma \rho_g^{(2)} = p_g^{(2)} - \frac{\gamma - 1}{2\gamma} p^{(1)2}, \tag{33b}$$

$$\gamma V u_g^{(2)} = p_g^{(2)} + \int_{-\infty}^{\xi} \frac{\partial p^{(1)}}{\partial \tau} d\xi - \frac{1}{2\gamma}\left(1 - \frac{1}{V^2}\right) p^{(1)2}, \tag{33c}$$

$$\gamma v_g^{(2)} = \left(1 - \frac{1}{V^2}\right)^2 \left(\frac{h^3}{3} - \frac{hy^2}{2} + \frac{y^3}{6}\right)\frac{\partial^3 p^{(1)}}{\partial \xi^3}$$
$$+ (h-y)\left\{\left(\frac{1}{V^2} - 1\right)\frac{\partial F_g}{\partial \xi} + \left(\frac{1}{V^2} + 1\right)\frac{\partial p^{(1)}}{\partial \tau} + \left(\frac{1}{\gamma V^4} + 1\right) p^{(1)}\frac{\partial p^{(1)}}{\partial \xi}\right\}, \tag{33d}$$

$$p_l^{(2)} = -\left(\frac{1}{V^2}-\frac{1}{c_*^2}\right)\left\{(1-h)y+\frac{y^2}{2}\right\}\frac{\partial^2 p^{(1)}}{\partial \xi^2}+F_l(\tau,\xi), \tag{34a}$$

$$\gamma\rho_* c_*^2 \rho_l^{(2)} = p_l^{(2)} - \frac{\gamma\rho_* c_*^2-1}{2\gamma\rho_* c_*^2}p^{(1)2}, \tag{34b}$$

$$\gamma\rho_* V u_l^{(2)} = p_l^{(2)} + \int_{-\infty}^{\xi}\frac{\partial p^{(1)}}{\partial\tau}d\xi + \frac{1}{2\gamma\rho_*}\left(\frac{1}{V^2}-\frac{1}{c_*^2}\right)p^{(1)2}, \tag{34c}$$

$$\gamma\rho_* v_l^{(2)} = -\left(\frac{1}{V^2}-\frac{1}{c_*^2}\right)^2\left\{\frac{(1-h)^3}{3}-\frac{(1-h)y^2}{2}-\frac{y^3}{6}\right\}\frac{\partial^3 p^{(1)}}{\partial\xi^3}$$
$$-(1-h+y)\left\{\left(\frac{1}{V^2}-\frac{1}{c_*^2}\right)\frac{\partial F_l}{\partial\xi}+\left(\frac{1}{V^2}+\frac{1}{c_*^2}\right)\frac{\partial p^{(1)}}{\partial\tau}+\left(\frac{1}{\gamma\rho_* V^4}+\frac{1}{c_*^2}\right)p^{(1)}\frac{\partial p^{(1)}}{\partial\xi}\right\}. \tag{34d}$$

By using (33) and (34) in (28*b*) we obtain the following consistency relations:

$$F_g(\tau,\xi) = F_l(\tau,\xi), \tag{35}$$

$$\frac{\partial p^{(1)}}{\partial\tau}+\frac{1}{2}\left(1+\frac{2}{\gamma V^2}\frac{h+(1-h)/\rho_*^2}{h+(1-h)/\rho_*}-\frac{1}{\gamma}\frac{h+(1-h)/\rho_*^2 c_*^2}{h+(1-h)/\rho_*}\right)p^{(1)}\frac{\partial p^{(1)}}{\partial\xi}$$
$$+\frac{\mu^4 h^2(1-h)^2}{6\rho_*}\frac{1-h+h/\rho_*}{[h+(1-h)/\rho_*]^3}V^2\frac{\partial^3 p^{(1)}}{\partial\xi^3}=0, \tag{36}$$

where $\mu^2 = 1-c_*^{-2}$. Apparently (36) shows that $p^{(1)}(\tau,\xi)$ is the solution of the Korteweg–de Vries equation. The coefficient of the nonlinear term is always of order unity. On the other hand, the dispersion parameter (the coefficient of $\partial^3 p^{(1)}/\partial\xi^3$) is small for atmospheric conditions and a moderate value of h, but it may become large with decreasing ρ_* and h, that is, for high-pressure conditions and a thin gaseous layer.

At present we can get a lot of information about the properties of the solution to the Korteweg–de Vries equation (e.g. Leibovich & Seebass 1974), and the above results suggest that observation of solitons is possible also for a pressure wave through a simple separated gas–liquid system.

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